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Relativistic Interactions Without Fields or Potentials

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Employing Poincaré degrees of freedom $M^{jk} = (\bar{K}, \bar{J})$ and $p^k = (E, \bar{p})$ transforming linearly (but inhomogeneously) under the action of the Poincaré group we define a number of quantities which we later identify with physical observables. The identifications are consistent with the nonrelativistic limit and with other requirements following from the Poincaré covariance. Next, we treat a free relativistic particle as composed of two interacting parts. Relativistic quantum commutation relations for their Poincaré algebras and a kind of (inverse) relativistic correspondence principle are used to generate (quasi-) classical equations of their relative motion. A simple example based on these ideas is explicitly solved.

1. INTRODUCTION

In this paper we intend to discuss kinematics and dynamics of relativistic (quasi-) classical systems. Any such particle[1] (system) is characterized by its degrees of freedom. Position coordinates associated with a particle are usually assumed to constitute the three-vector part of a four-vector in a preexisting Minkowski space. The remaining degrees of freedom such as the total angular momentum $\bar{J}$, the linear momentum $\bar{p}$, the energy $\bar{E}$, etc., are introduced separately. The connection between these, so to speak, internal degrees of freedom and a position vector $\bar{R}$ associated with the particle is imposed by the relation

$$\bar{u} \overset{\text{def}}{=} \frac{c^2\bar{p}}{E} = \frac{d\bar{R}}{dT} \tag{1.1}$$

[1]In this paper we use the word "particle" in a very general sense. It stands for any physical system which can be given an identity.

where $(cT, \bar{R})$ constitutes a "position" four-vector in the preexisting Minkowski space.

In our approach here we intend to proceed somewhat differently. A massive particle will here be described entirely in terms of ten Poincaré covariant degrees of freedom. The parameters representing these degrees of freedom will be identified with the components of the four-momentum vector $P^j=(E, \bar{p})$ and the four-angular momentum tensor $M^{kl}=(\bar{K}, \bar{J})$. These quantities written out in the four-vector form have the advantage of transforming linearly (but inhomogeneously) under the action of the Poincaré group. This is the reason they are so appropriate for explicit calculations. As we shall see in the next section, some of these quantities do not even need to have any direct physical meaning. Most of the "real" physical observables become, according to our definitions and identifications, derived objects constructed out of the ten Poincaré covariant degrees of freedom represented by P^j and M^{kl}. One of the consequences of our approach is that the second equality in (1.1) cannot be valid in general. We obtain a new concept which we call the speed and which does not always coincide with the notion of velocity as defined in (1.1).

In the third section using quantum mechanical commutation relations for the Poincaré algebra and a relativistic version of the correspondence principle we derive equations of motion for two interacting inseparable particles forming a closed composite system. No fields are involved in the description. Finally, utilizing what we call our dynamical principle we solve a very simple example (Bette, 1980). We wish also to emphasize that the main result (the dynamical relativistic principle) is the same as in our previous paper (Bette, 1980). The way of reasoning is, however, somewhat different and we hope clearer. The identifications of physical observables are also slightly changed. From the nonrelativistic point of view these changes do no matter but for the internal consistency of the exposition they seem to be necessary.

2. CONNECTION BETWEEN POINCARÉ DEGREES OF FREEDOM AND PHYSICAL OBSERVABLES: NOTATIONS, RELATIVISTIC FORMULAS, DEFINITIONS, AND IDENTIFICATIONS

In order to fix ideas we introduce Poincaré degrees of freedom (Pdf) at the very beginning. These are represented by two polar vectors $\bar{p}$, $\bar{K}$, an axial vector $\bar{J}$, and a scalar E. Using four-tensor notation we may also write

$$M_{kl}=(\bar{K}, \bar{J}), \qquad P_j=(E, \bar{p}) \tag{2.1}$$

For visualization we shall, however, often use the three-vector notation. The action of the Poincaré group on the introduced quantities is specified by the following rules.

Rotation:

$$\bar{K}'=\bar{K}+\left(1+\tfrac{1}{4}n^2\right)^{-1}\left[\bar{n}\times\left(\bar{K}+\tfrac{1}{2}\bar{n}\times\bar{K}\right)\right] \tag{2.2}$$

$$\bar{J}'=\bar{J}+\left(1+\tfrac{1}{4}n^2\right)^{-1}\left[\bar{n}\times\left(\bar{J}+\tfrac{1}{2}\bar{n}\times\bar{J}\right)\right] \tag{2.3}$$

$$\bar{p}'=\bar{p}+\left(1+\tfrac{1}{4}n^2\right)^{-1}\left[\bar{n}\times\left(\bar{p}+\tfrac{1}{2}\bar{n}\times\bar{p}\right)\right] \tag{2.4}$$

$$E'=E \tag{2.4a}$$

with $\bar{n}=2\hat{n}\tan(\alpha/2)$ being an axial rotation vector $\hat{n}\cdot\hat{n}=1$, $n=2\tan(\alpha/2)$ and with α denoting the angle of rotation.

Alternatively we may also write

$$\bar{R}'=\bar{R}\cos\alpha+(\hat{n}\times\bar{R})\sin\alpha+2\hat{n}(\hat{n}\cdot\bar{R})\sin^2(\alpha/2) \tag{2.5}$$

where $\bar{R}$ symbolizes any of $\bar{p}$, $\bar{K}$, $\bar{J}$ above.

Boost:

$$\bar{K}'=\gamma\left\{\bar{K}+(\bar{J}\times\bar{v})/c-\left[\gamma/(1+\gamma)c^2\right]\bar{v}(\bar{v}\cdot\bar{K})\right\} \tag{2.6}$$

$$\bar{J}'=\gamma\left\{\bar{J}-(\bar{K}\times\bar{v})/c-\left[\gamma/(1+\gamma)c^2\right]\bar{v}(\bar{v}\cdot\bar{J})\right\} \tag{2.7}$$

$$\bar{p}'=\bar{p}+\bar{v}\left[\frac{\bar{v}\cdot\bar{p}}{v^2}(\gamma-1)-\frac{\gamma}{c^2}E\right] \tag{2.8}$$

$$E'=\gamma(E-\bar{v}\cdot\bar{p}) \tag{2.9}$$

with $\gamma=(1-v^2/c^2)^{-1/2}$, $\bar{v}\cdot\bar{v}=v^2$ and $\hat{v}=\bar{v}/v$.

Space Translation:

$$\bar{K}'=\bar{K}+\bar{A}E/c \tag{2.10}$$

$$\bar{J}'=\bar{J}+(\bar{A}\times\bar{p}) \tag{2.11}$$

$$\bar{p}'=\bar{p} \tag{2.12}$$

$$E'=E \tag{2.13}$$

with $\bar{A}$ being a translation vector.

Time Translation:

$$\bar{K}' = \bar{K} + ct\bar{p} \tag{2.14}$$

$$\bar{J}' = \bar{J} \tag{2.15}$$

$$\bar{p}' = \bar{p} \tag{2.16}$$

$$E' = E \tag{2.17}$$

with t representing the lapse of (external) time. Transformation parameters are $\bar{v}$ (a polar vector), $\bar{n}$ (an axial vector), $\bar{A}$ (a polar vector), and ct (a scalar)—ten parameters altogether, i.e., as many as Pdf.

For later use we construct following vectors and scalars out of Pdf:

$$\bar{R} \stackrel{\text{def}}{=} c\bar{K}/E \tag{2.18}$$

$$\bar{L} \stackrel{\text{def}}{=} \bar{R} \times \bar{p} = c\bar{K} \times \bar{p}/E \tag{2.19}$$

$$\bar{u} \stackrel{\text{def}}{=} c^2\bar{p}/E \tag{2.20}$$

$$\bar{S} \stackrel{\text{def}}{=} \frac{E}{mc^2}(\bar{J} - \bar{L}) \tag{2.21}$$

$$S_0 \stackrel{\text{def}}{=} (\bar{J} \cdot \bar{p})/mc = c(\bar{S} \cdot \bar{p})/E \tag{2.22}$$

with $m = (E^2/c^4 - p^2/c^2)^{1/2}$.

It can easily be seen that S_0 and $\bar{S}$ are not affected by space or time translations whereas under boosts and rotations they transform like E and $\bar{p}$, respectively.

More definitions:

$$\bar{e} \stackrel{\text{def}}{=} \bar{K} - \frac{E^2}{m^2c^4}\bar{K} + (\bar{p} \times \bar{J})\frac{E}{m^2c^3} + \frac{\bar{p}(\bar{p} \cdot \bar{K})}{m^2c^2} \tag{2.23}$$

$$\bar{h} \stackrel{\text{def}}{=} \bar{J} + \frac{p^2}{m^2c^2}\bar{J} + (\bar{p} \times \bar{K})\frac{E}{m^2c^3} - \frac{\bar{p}(\bar{p} \cdot \bar{J})}{m^2c^2} \tag{2.24}$$

It may even here easily be seen that $\bar{e}$, $\bar{h}$ are not affected by space or time translations whereas under boosts and rotations they transform like $\bar{K}$ and $\bar{J}$, respectively.

From (2.21) we see that $\bar{J}$ may be decomposed into an "orbital" and an "intrinsic" part in a Poincaré covariant way according to

$$\bar{J}=(1-c^2p^2/E^2)^{1/2}\bar{S}+\bar{L} \tag{2.25}$$

Inserting (2.19) and (2.25) into (2.23) and (2.24) we get

$$\bar{e}=\frac{1}{mc}(\bar{p}\times\bar{S}) \tag{2.26}$$

$$\bar{h}=\frac{E}{mc^2}\left[\bar{S}-\frac{c^2\bar{p}}{E^2}(\bar{S}\cdot\bar{p})\right] \tag{2.27}$$

In the special case when $m\to 0$ from (2.22), (2.25), (2.26), and (2.27) we obtain

$$M^{ij}=(\bar{K},\bar{L})=\left(\bar{K},c\{\bar{K}\times\bar{p}\}/E\right) \tag{2.28}$$

$$\bar{p}\times\bar{S}=0 \tag{2.29}$$

$$S_0=\frac{c}{E}(\bar{S}\cdot\bar{p})=s \qquad \text{(1/2 twistor norm)} \tag{2.30}$$

$$\bar{S}=\frac{c^2\bar{p}}{E^2}(\bar{S}\cdot\bar{p})=s\frac{c\bar{p}}{E} \tag{2.31}$$

Note that in this case we have $S_0S_0-\bar{S}\cdot\bar{S}=S_kS^k=0$. Notice also that (massless case) the degrees of freedom represented by $\bar{S},\bar{e},\bar{h}$ and introduced by means of our original Pdf represented by $\bar{J}$, $\bar{K}$, $\bar{p}$, and E (massive case) now decouple from their definitions forming completely new and independent quantities.

The concept of a massless spinning classical particle obtained as a limiting case of a massive spinning relativistic particle in the way indicated, i.e., the concept of a massless classical particle with degrees of freedom given by $\bar{K},\bar{p}$, and s has been previously introduced by Penrose (1968, 1972) in the context of twistor theory. Note that s is a Poincaré-invariant pseudoscalar. It is also invariant with respect to conformal transformations [which we do not discuss in this paper (Penrose, 1968)] of $\bar{K}$ and $\bar{p}$ ($m=0$).

We make now the following identifications: We identify $\bar{p}$ with the linear momentum of a free particle, E with the energy of a free particle, $\bar{J}$ with the total angular momentum of a free particle, $\bar{L}$ with the orbital angular momentum of a free particle, $\bar{S}$ with the internal angular momentum of a free particle,

$$\frac{c}{E}\bar{K}'\equiv\bar{r}=c\bar{K}/E+tc^2\bar{p}/E=\bar{R}+t\bar{u}$$

with the position vector of a free particle, and m with the inertial mass of a free particle.

Notice that here we have $d\bar{r}/dt=\bar{u}$, which is quite similar to (1.1). As a digression we here mention that throughout this paper for explicit calculations we have utilized the four-tensor notation with the following conventions:

$$M^{\alpha 0}=M_{0\alpha}=-M_{\alpha 0}=-M^{0\alpha}=K_\alpha, \qquad \bar{K}=(K_1,K_2,K_3)$$

$$M^{\alpha\beta}=M_{\alpha\beta}=\varepsilon_{\alpha\beta\sigma}J_\sigma, \bar{J}=(J_1,J_2,J_3)$$

$$-P^\alpha=P_\alpha=P_\alpha, \qquad P^0=P_0=E/c, \bar{p}=(p_1,p_2,p_3) \tag{2.32}$$

$$\varepsilon_{[\alpha\beta\sigma]}=\varepsilon_{\alpha\beta\sigma}, \qquad \varepsilon_{321}=1, \qquad P^iP_i=m^2c^2$$

$$g_{00}=g^{00}=-g^{\alpha\alpha}=-g_{\alpha\alpha}=1, \qquad g^{ik}=0 \quad \text{if } i\neq k \not\Sigma$$

Square brackets around the sub- or superscripts denote antisymmetrization and summation convention is assumed throughout. A massive free particle characterized by the Pdf as discussed above defines its own proper (internal) time τ. We introduce a typical intrinsic time scale associated with such a free particle:

$$\tau_0 \overset{\text{def}}{=} 2\frac{(1+s^2)\hbar}{mc^2} \tag{2.33}$$

where $s^2=\bar{S}\cdot\bar{S}-S_0S_0=-S^iS_i$ and $m=(E^2/c^4-p^2/c^2)^{1/2}$, with $\hbar$ Planck's constant.

Let us now assume that a free particle may in its own rest frame with respect to its proper time and relative its own center of energy be regarded as composed of (in the simplest case) two interacting parts:

$$\bar{0}=\bar{K}=\bar{K}_1(\tau)+\bar{K}_2(\tau)$$

$$\bar{0}=\bar{p}=\bar{p}_1(\tau)+\bar{p}_2(\tau)$$

$$\bar{S}=\bar{J}=\bar{J}_1(\tau)+\bar{J}_2(\tau)$$

$$mc^2=E=E_1(\tau)+E_2(\tau) \tag{2.34}$$

The splitting as it stands above is completely arbitrary, i.e., it is a functional of ten arbitrary functions of τ. Nevertheless the decomposition is covariant with respect to the Poincaré group because in the four-tensor notation using the linearity of such a representation we may write

$$M^{jk}=M^{jk}_{(1)}(\tau)+M^{jk}_{(2)}(\tau)$$

$$P^k=P^k_{(1)}(\tau)+P^k_{(2)}(\tau) \tag{2.35}$$

In order to unambiguously (for a given $\bar{S}$ and m) evaluate the ten functions of τ in the linear splitting above we need some kind of a dynamical principle. This will be given in the next section. First let us make the following (compare the free-particle case) identifications: We identify $\bar{p}_i$ with the instantaneous linear momentum vector of the ith interacting part of the composite free particle relative to the rest frame of the latter; E_i with the instantaneous energy of the ith interacting part of the composite free particle relative to the rest frame of the latter; $\bar{J}_i$ with the instantaneous total angular momentum vector of the ith interacting part relative to the center of energy of the total system in its own rest frame; $\bar{r}_i \overset{\text{def}}{=} c\bar{K}_i/E_i$ with the instantaneous position vector of the ith interacting part relative to the center of energy of the total system in its own rest frame; $\bar{L}_i \overset{\text{def}}{=} \bar{r}_i \times p_i$ with the instantaneous orbital angular momentum vector of the ith interacting part relative to the center of energy of the total system in its own rest frame; $\bar{S}_i \overset{\text{def}}{=} \dfrac{E_i}{m_i c^2}(\bar{J}_i - \bar{L}_i)$ with the instantaneous intrinsic angular momentum vector of the ith interacting part of the composite free particle relative to the rest frame of the latter. Here $i=1,2$. The term "instantaneous" refers to the proper (intrinsic) time associated with the total system.

Note that the instantaneous (inertial velocity) vector

$$\bar{u}_i = c^2\bar{p}_i/E_i \tag{2.36}$$

is not in general equal to

$$\bar{w}_i \overset{\text{def}}{=} \frac{d\bar{r}_i}{d\tau} \tag{2.37}$$

The vector $\bar{w}_i$ will in the sequel be called the speed vector of the ith interacting part of the composite system.

3. THE DYNAMICAL PRINCIPLE[2]

In relativistic quantum physics the Pdf associated with a free particle become operators obeying the very well known rules for the Poincaré

[2]In this section in order to simplify computations we sometimes put $h=1$ and $c=1$.

algebra:

$$[\hat{P}^i, \hat{P}^k]=0, \qquad [\hat{M}^{ij}, \hat{P}^k]=2ig^{k[j}\hat{P}^{i]}$$

$$[\hat{M}^{ij}, \hat{M}^{lk}]=2i\left(g^{l[j}\hat{M}^{i]k}+g^{k[i}\hat{M}^{j]l}\right) \tag{3.1}$$

The splitting in (2.34) is now given by

$$\hat{P}^i=\hat{P}^i_{(1)}(\tau)+\hat{P}^i_{(2)}(\tau), \qquad \hat{M}^{ik}=\hat{M}^{ik}_{(1)}(\tau)+\hat{M}^{ik}_{(2)}(\tau) \tag{3.2}$$

where τ is the proper time parameter of the free composite system. Further, we assume here that the "Pdf" operators corresponding to the ith interacting part of the composite free particle, obtained by the splitting of the "Pdf" operators of the latter according to (3.2), also obey the commutation rules in (3.1) for every value of the proper (intrinsic) time and that "Pdf" operators corresponding to the first interacting part commute with the "Pdf" operators corresponding to the second interacting part.

We recall that the squared mass and squared spin operators of a free (composite or simple) particle commute with all operators in its Poincaré algebra, i.e., with all "Pdf" operators obeying (3.1). They are explicitly given by

$$\hat{m}^2=\hat{P}^i\hat{P}_i/m^2$$

$$\hat{s}^2=-\hat{S}^i\hat{S}_i=-(1/4)\varepsilon_{ijkl}\varepsilon^i_{mnr}\hat{P}^r\hat{M}^{mn}\hat{P}^l\hat{M}^{jk}/m^2 \tag{3.3}$$

where m^2 is the eigenvalue of $\hat{P}^i\hat{P}_i$.

We assume that there always exists a Poincaré-invariant Hamilton operator associated with a free (possibly composite) particle and with its proper (intrinsic) time. The Hamilton operator is then always a function of the operators in (3.3):

$$\hat{H}=\hat{H}(\hat{m}^2, \hat{s}^2) \tag{3.4}$$

Imitating nonrelativistic dynamics we obtain relativistic quantum mechanical equations of motion for operators describing the two inseperable interacting parts forming the free composite system (Bette, 1980):

$$\tau_0\frac{d\hat{P}^i_{(a)}}{d\tau}=-i[\hat{H}, \hat{P}^i_{(a)}], \qquad \tau_0\frac{d\hat{M}^{ik}_{(a)}}{d\tau}=-i[\hat{H}, \hat{M}^{ik}_{(a)}]$$

$$(a)=(1),(2) \tag{3.5}$$

τ_0 is a time scale introduced in (2.33) and s^2 is a fixed eigenvalue of the operator $\hat{s}^2$ in (3.3). τ denotes the lapse of the proper (internal) time of the total particle.

Let us choose a very simple form for the "Hamiltonian" in (3.4) (Bette, 1980):

$$\hat{H}=\hat{m}^2+\hat{s}^2 \tag{3.6}$$

Performing the commutations on the right-hand side of (3.5), neglecting terms of order $\hbar^2$, and letting operators become the usual (c) numbers we get the following set of (quasi) classical dynamical equations of motion:

$$\begin{aligned}
\tau_0\frac{dP^a_{(1)}}{d\tau}&=2S^i\varepsilon_{ijkl}P^l g^{a[k_P j]}_{(1)}/m\\
\tau_0\frac{dM^{ab}_{(1)}}{d\tau}&=\frac{4}{m^2}P_j g^{j[a_P b]}_{(1)}+2S^i\varepsilon_{ijkl}M^{jk}g^{l[a_P b]}_{(1)}/m\\
&\quad+2S^i\varepsilon_{ijkl}P^l\left(g^{a[k_M j]b}_{(1)}+g^{b[j_M k]a}_{(1)}\right)/m\\
P^a_{(2)}&=P^a-P^a_{(1)},\qquad M^{ab}_{(2)}=M^{ab}-M^{ab}_{(1)}
\end{aligned} \tag{3.7}$$

The procedure as described above constitutes our dynamical principle as referred to in the last section and in the Introduction. The dynamical principle fixes the splitting of the free composite particle. The decomposition is not arbitrary any more, depending entirely on the form of the Hamilton operator in (3.4).

If we place an orthonormal space tetrad at the center of energy of the total system in its own rest frame—i.e., if we put

$$\begin{aligned}
&M^{21}=M^{13}=M^{\alpha 0}=0,\qquad M^{32}=s,\text{i.e., } S_0=0 \text{ and } \bar{S}=(s,0,0),\\
&P_0=m,\qquad \bar{p}=\bar{0}
\end{aligned} \tag{3.8}$$

then the equations in (3.7) become

$$\begin{aligned}
&\frac{dP^0_{(1)}}{d\tau}=0,\ \frac{dP^1_{(1)}}{d\tau}=0,\ \tau_0\frac{dP^2_{(1)}}{d\tau}=-2sP^3_{(1)},\ \tau_0\frac{dP^3_{(1)}}{d\tau}=2sP^2_{(1)}\\
&\tau_0\frac{dM^{10}_{(1)}}{d\tau}=-\frac{2}{m}P^1_{(1)},\ \tau_0\frac{dM^{20}_{(1)}}{d\tau}=2\frac{-1-s^2}{m}P^2_{(1)}+2sM^{30}_{(1)}\\
&\tau_0\frac{dM^{30}_{(1)}}{d\tau}=2\frac{-1-s^2}{m}P^3_{(1)}-2sM^{20}_{(1)}\\
&\frac{dM^{\alpha\beta}_{(1)}}{d\tau}=0
\end{aligned} \tag{3.9}$$

Putting

$$\omega_0 = 2s/\tau_0, \qquad -P^{\alpha}_{(1)} = p_\alpha = \mu\gamma u_\alpha, \qquad J_\alpha = \tfrac{1}{2}\varepsilon_{\alpha\beta\sigma} M^{\beta\sigma}_{(1)}$$

$$K_\alpha = M^{\alpha 0}_{(1)}, \qquad P^0_{(1)} = E = \mu\gamma \tag{3.10}$$

where μ is the rest mass of particle number 1 and $\gamma = (1-u^2)^{-1/2}$ we may rewrite (3.9) in a more transparent way:

$$\dot{E} = 0, \qquad \dot{p}_1 = 0, \qquad \dot{\bar{J}} = \bar{0}, \qquad \dot{K}_1 = \frac{2}{m\tau_0} p_1$$

$$\dot{K}_2 = p_2 + \omega_0 K_3, \qquad \dot{p}_2 = -\omega_0 p_3$$

$$\dot{K}_3 = p_3 - \omega_0 K_2, \qquad \dot{p}_3 = +\omega_0 p_2 \tag{3.11}$$

with the dot denoting derivation with respect to the proper time. The solutions of (3.11) are

$$p_2 = A\cos\omega_0\tau - B\sin\omega_0\tau$$

$$p_3 = A\sin\omega_0\tau + B\cos\omega_0\tau$$

$$K_2 = \left(A\frac{1+s^2}{ms} + D\right)\sin\omega_0\tau + \left(B\frac{1+s^2}{ms} - C\right)\cos\omega_0\tau$$

$$K_3 = C\sin\omega_0\tau + D\cos\omega_0\tau \tag{3.12}$$

$$p_1 = \text{const}$$

$$K_1 = \frac{2}{m\tau_0} p_1\tau + F$$

$$E = \text{const}$$

$$\bar{J} = \text{const}$$

Choosing initial values in a suitable way the solutions above become

$$p_2 = p\cos\omega_0\tau, \qquad p_3 = p\sin\omega_0\tau$$

$$K_2 = \left(p\frac{1+s^2}{ms} + D\right)\sin\omega_0\tau, \qquad K_3 = D\cos\omega_0\tau$$

$$p_1 = 0, \qquad K_1 = 0, \qquad J_3 = 0, \qquad J_2 = 0, \qquad J_1 = \sigma \tag{3.13}$$

$$E = (\mu^2 + p^2)^{1/2}$$

$$p = (p_1^2 + p_2^2)^{1/2}$$

In case $s \gg 1, u_2 \gg \omega_0 l_3, u_3 \gg \omega_0 l_2$ [where $l_i \quad i=2,3$ characterize typical distances between the center of energy of the total composite system in its own rest frame and the ith interacting part and where u_i denote components of the inertial velocities as defined in (2.36)] we note from (3.11) that the speed and the velocity vectors coincide, i.e., we have

$$\dot{K}_2/E_2 \cong u_2, \qquad \dot{K}_3/E_3 \cong u_3, \qquad \dot{K}_1/E_1 \cong 0 \tag{3.14}$$

If in addition $u_i \ll c$ than we get from (3.11) a very simple approximative solution according to which one of the (sub)particles performs a spiral motion squirming inwards, while the other is spiralling outwards in the opposite direction.

The general case is—because of the existence of the two velocity concepts (inertial velocity and noninertial speed)—more intricate and will be discussed more fully elsewhere.

4. CONCLUSIONS AND REMARKS

It is more natural to study the dynamical principle formulated in this paper using the theory of twistors (Penrose, 1968, 1972, 1975; Hughston, 1979). Some difficulties appear, however. The twistor description increases the number of degrees of freedom due to the underlying conformal symmetry. These additional degrees of freedom related to the "internal" symmetries encountered in elementary particle physics are hard to interpret on the classical level. On the other hand, the possibility arises of treating a massive spinning free particle as composed of two (bounded) massless interacting parts. The procedure in this case is rather straightforward. In a forthcoming paper we intend to present such calculations.

NOTE ADDED IN PROOF

The parameter in our "equations of motion" cannot, by physical reasons, be identified with the proper time of the total system precisely because of (2.36) and (2.37). The parameter constitutes only an (as yet unknown) function of the proper time t. t may, in fact, be determined as a function of τ if we require

$$|\bar{w}_i| \overset{\text{def}}{=} \left|\frac{d\bar{r}_i}{d\tau}\right| \frac{d\tau}{dt} = \frac{c^2 |\bar{p}_i|}{E_i}$$

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